# Relativistic Kinematics XXXXX, B 2.7

In our applications of 2-body scattering, we will almost always consider a frame where the target is at rest, and a projectile strikes it, leading to a scattered projectile and recoiling target.

Also in relativistic kinematics, we also introduce a new type of particle, the "massless particle". e.g. a photon, a neutrino (nearly massless!)

## Kinematic quantities in a relativistic formalism

| | particle with mass m | | massless particle |
|---|---|---|---|
| | Rel | Non Rel | |
| Rest mass energy | $mc^2$ | N/A | 0 |
| Total energy ($E$) | $mc^2\gamma$ | N/A | $E$ |
| Kinetic energy (Total − Rest) ($K, T$) | $mc^2(\gamma-1)$ | $\frac{1}{2}mv^2$ | $E$ |
| Momentum ($\vec{p}$) | $m\vec{v}\gamma$ | $m\vec{v}$ | $(E/c)\hat{n}$ |

direction of motion

## Relativistic $\gamma, \beta$ factors

Definitions $\vec{\beta} = \vec{v}/c$ $c$ = speed of light = 299,792,458 m/s (exactly)

$\gamma = (1-\beta^2)^{-1/2}$

as a consequence $\gamma^2(1-\beta^2) = 1$ (useful property) (memorize this!)

The concepts of "total" energy and "rest mass" energy were introduced by Einstein to describe relativistic physics. They have no meaning in non-relativistic physics.

In the limit that $\beta \to 0$, i.e. $|\vec{v}| \ll c$ for particle with mass, we do this by casting the results in terms of kinetic energy or momentum, and then keep the first non-vanishing result. Always. We do this through the process of Taylor (Maclaurin) expansions. (See text on the next pages). The next term in the series, is identified as the lowest order relativistic correction. For example, it can be shown that

Kinetic energy of a particle with mass

$$K = mc^2(\gamma - 1)$$

$$= mc^2\left[(1-\beta^2)^{-1/2} - 1\right]$$

$$\overset{\beta \to 0}{\rightsquigarrow} mc^2\left[1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + O(\beta^6) - 1\right]$$

$$= \frac{mc^2\beta^2}{2}\left(1 + \frac{3}{4}\beta^2 + O(\beta^4)\right)$$

$$= \frac{1}{2}mv^2\left(1 + \frac{3}{4}\beta^2 + O(\beta^4)\right)$$

So, we have identified

$K_{NR} = \frac{1}{2}mv^2$ the familiar NR result

and its lowest order correction $+\frac{3}{4}\beta^2$ ← relative

So, if $\beta = 0.1$ , 1/10 the speed of light

correction is 0.75% only!

For momentum

$|\vec{p}| = p = m v \gamma$

$\xrightarrow[\beta \ll 0]{} m v \left(1 + \frac{\beta^2}{2}\right)$

$mv$ familiar NR result

1ST order R correction $\frac{1}{2}\beta^2$

An example of where a Taylor series approach is necessary.

If $\frac{K}{mc^2} = 1.00 \times 10^{20}$,

how close is $v$ to $c$?

i.e. What is $\frac{c-v}{c}$ ?

Recall $K = mc^2(\gamma - 1)$

let $r = \frac{K}{mc^2}$

$\therefore \quad \gamma - 1 = r$

$$\gamma = 1 + r$$

$$\frac{1}{(1-\beta^2)^{1/2}} = 1 + r$$

$$1 - \beta^2 = \frac{1}{(1+r)^2}$$

$$\beta^2 = 1 - \frac{1}{(1+r)^2}$$

$$= \frac{(1+r)^2 - 1}{(1+r)^2} = \frac{2r + r^2}{(1+r)^2}$$

$$\beta = \frac{\sqrt{2r+r^2}}{1+r}$$

$$\frac{c-v}{c} = 1-\beta$$

$$1-\beta = 1 - \frac{\sqrt{2r+r^2}}{1+r}$$

$$= \frac{1+r-\sqrt{2r+r^2}}{1+r}$$

$$= \frac{1+\frac{1}{r}-\sqrt{1+\frac{2}{r}}}{\left(1+\frac{1}{r}\right)}$$

$$= \frac{1+\frac{1}{r}-\left(1+\frac{1}{r}-\frac{1}{2r^2}+O\left(\frac{1}{r^3}\right)\right)}{1+\frac{1}{r}}$$

$$= \frac{\frac{1}{2r^2}+O\left(\frac{1}{r^3}\right)}{1+\frac{1}{r}}$$

$$= \frac{1}{2r^2}+O\left(\frac{1}{r^3}\right)$$

$\uparrow$ $5\times10^{-41}$ $\qquad$ $\uparrow$ $O(10^{-60})$

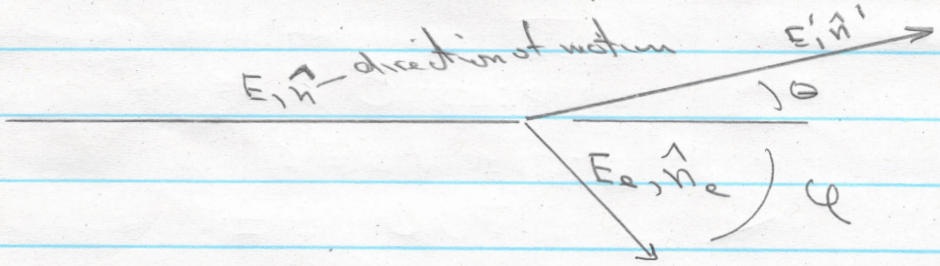
Ans To within 5 parts in $10^{41}$ !

Can not do this with any calculator!

# Examples of relativistic kinematics

A photon with energy $E$, strikes an electron that is at rest. After the collision, what is the relationship between $E'$, the scattered photon energy, and its scattering angle?

[Aside. This is the "most famous" relativistic kinematic relationship in all of physics.]




CoE ⇒ $$E + mc^2 = E' + mc^2\gamma \quad (1)$$

CoM $$\frac{E\hat{n}}{c} = \frac{E'\hat{n}'}{c} + m\vec{v}_e\gamma \quad (2)$$

$$\gamma = \left[1 - \frac{\vec{v}_e^{\,2}}{c^2}\right]^{-1/2}$$

(1)/$mc^2$, put $e^-$ on RHS $$\alpha + 1 - \alpha' = \gamma \quad (3)$$

(2)/$mc$, put $e^-$ on RHS $$\alpha\hat{n} - \alpha'\hat{n}' = \vec{\beta}\gamma \quad (4)$$

$\alpha \equiv E/mc^2$, $\alpha' = E'/mc^2$

$\vec{\beta} = \vec{v}_e/c$

Want to see $\gamma^2 - \beta^2\gamma^2 = 1$, i.e.

$$(3)^2 - |(4)|^2 \Rightarrow$$

$$(\alpha + 1 - \alpha')^2 - |\alpha\hat{n} - \alpha'\hat{n}'|^2 = \underbrace{\gamma^2 - \beta^2\gamma^2}_{=1}$$

expand

$$\alpha^2 + 1 + \alpha'^2 + 2\alpha - 2\alpha' - 2\alpha\alpha' - (\alpha^2 - 2\alpha\alpha'\hat{n}\cdot\hat{n}' + \alpha'^2) = 1$$

and $\div 2$ and reorganize

$$\alpha - \alpha' = \alpha\alpha' - \alpha\alpha'\underbrace{\hat{n}\cdot\hat{n}'}_{\cos\theta}$$

or, $\div \alpha\alpha'$

$$\boxed{\frac{1}{\alpha'} - \frac{1}{\alpha} = 1 - \cos\theta} \quad (5)$$

or $$\boxed{\frac{mc^2}{E'} - \frac{mc^2}{E} = 1 - \cos\theta} \quad (6)$$

Some interesting observations

1) $E'$ & $\theta$ are related to each other

2) If $\cos\theta = 1$ $(\theta = 0°)$ $\quad E' = E$

3) If $\cos\theta = -1$ $(\theta = \pi, 180°$ : photon backscatters)

$$\frac{1}{\alpha'} = 2 + \frac{1}{\alpha} \qquad \frac{1}{\alpha'} > 2 \Rightarrow \alpha' < \frac{1}{2}$$

(in the backdirection)

4) If $\cos\theta = -1$ & $\alpha \to \infty$

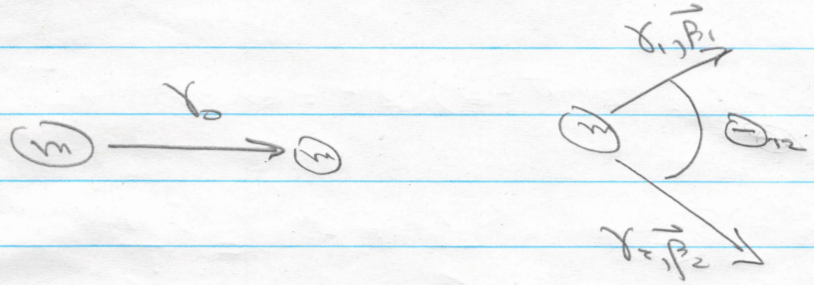
$$\alpha' = \frac{1}{2} \; ; \; E' = \frac{1}{2}mc^2 \; !$$

Problem for future assignment

Relate the KE of the recoil electron to its angle $\theta$. Draw conclusions!

## Another example

A relativistic billiard ball strikes another at rest. Relate the opening angle of the outgoing billiard balls to their kinetic energies.




$$\text{CoE:} \quad mc^2\gamma_0 + mc^2 = mc^2\gamma_1 + mc^2\gamma_2 \qquad (7)$$

$$\text{CoM} \quad mc\vec{\beta}_0\gamma_0 = mc\vec{\beta}_1\gamma_1 + mc\vec{\beta}_2\gamma_2 \qquad (8)$$

$\div$ (7) by $mc^2$, (8) by $mc$, put initial $e^-$ on LHS

$$\gamma_0 = \gamma_1 + \gamma_2 - 1 \qquad (9)$$

$$\vec{\beta}_0\gamma_0 = \vec{\beta}_1\gamma_1 + \vec{\beta}_2\gamma_2 \qquad (10)$$

$$(9)^2 - (10)^2 = \underbrace{\gamma_0^2 - \beta_0^2\gamma_0^2}_{=1} = (\gamma_1 + \gamma_2 - 1)^2 - |\vec{\beta}_1\gamma_1 + \vec{\beta}_2\gamma_2|^2$$

$$1 = \gamma_1^2 + \gamma_2^2 + 1 + 2\gamma_1\gamma_2 - 2\gamma_1 - 2\gamma_2 - (\beta_1^2\gamma_1^2 + \beta_2^2\gamma_2^2 + 2\beta_1\beta_2\gamma_1\gamma_2\cos\theta_{12})$$

$$0 = 2 + 2\gamma_1\gamma_2 - 2\gamma_1 - 2\gamma_2 - 2\beta_1\beta_2\gamma_1\gamma_2 \cos\Theta_{12}$$

$$\text{or} \quad \cos\Theta_{12} = \frac{1 + \gamma_1\gamma_2 - \gamma_1 - \gamma_2}{(\beta_1\gamma_1)(\beta_2\gamma_2)}$$

Let $\gamma_1 = k_1 + 1$ $\qquad K_1 = k_1 mc^2 = mc^2(\gamma_1 - 1)$

$K_2 =$ similar

also $\gamma^2 - \beta^2\gamma^2 = 1 = \beta^2\gamma^2 = \gamma^2 - 1 = (\gamma - 1)(\gamma + 1)$

$= k(k+2)$

[$k$ is a dimensionless form of the kinetic energy.]

So,

$$\cos\Theta_{12} = \frac{1 + (k_1+1)(k_2+1) - (k_1+1) - (k_2+1)}{[k_1(k_1+2)\,k_2(k_2+2)]^{1/2}}$$

$$= \frac{1 + k_1k_2 + k_1 + k_2 + 1 - k_1 - 1 - k_2 - 1}{k_1(k_1+2)(k_2)(k_2+2)}$$

$$\boxed{\cos\Theta_{12} = \left[\frac{k_1k_2}{(k_1+2)(k_2+2)}\right]^{1/2}} \quad (11)$$

$$\text{or} \quad \cos\Theta_{12} = \left[\frac{K_1K_2}{(K_1 + 2mc^2)(K_2 + 2mc^2)}\right]^{1/2} \quad (12)$$

## Discussion

1) If $k_1 << 1$ and $k_2 << 1$

$\cos\theta_{12} \to 0$ The classical result

2) If $k_1 << 1$ or $k_2 << 1$

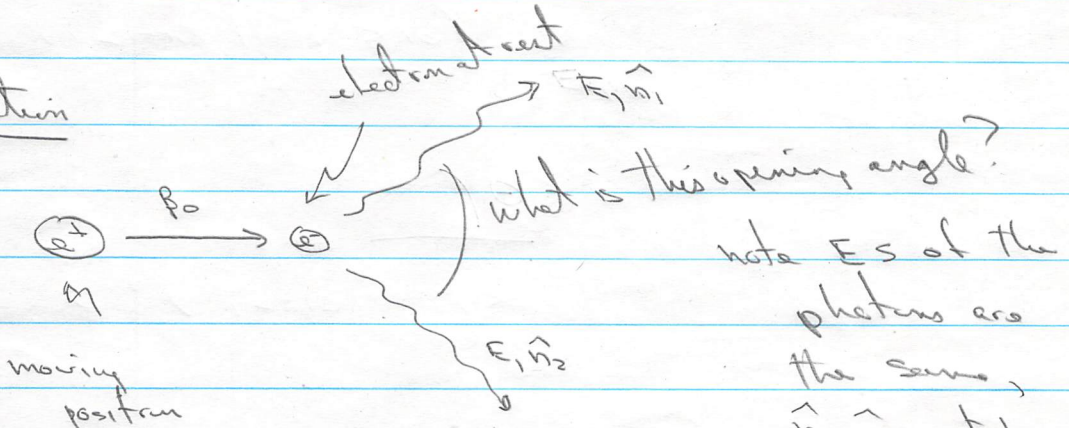
$\cos\theta_{12} \to 0$ (new result)

3) If $k_1 >> 1$ and $k_2 >> 1$

$\cos\theta_{12} \to 1$

interpretation? (forward directed)

## Annihilation




CoE $\quad mc^2\gamma_0 + mc^2 = 2E \qquad (1)$

CoM $\quad mc\vec{\beta}_0\gamma_0 = E(\hat{n}_1 + \hat{n}_2) \qquad (2)$

$(1)/mc^2 \quad E/mc^2 \equiv \alpha \qquad \gamma_0 + 1 = 2\alpha \qquad (3)$

$(2)/mc \qquad \vec{\beta}\gamma_0 = \alpha(\hat{n}_1 + \hat{n}_2) \qquad (4)$

$(3)^2 - |(4)|^2 \Rightarrow \quad (\gamma_0+1)^2 - \beta^2\gamma_0^2 = 4\alpha^2 - \alpha^2(2 + 2\hat{n}_1\cdot\hat{n}_2)$

$\gamma_0^2 + 2\gamma_0 + 1 - \beta^2\gamma_0^2 = 2\alpha^2(1 - \hat{n}_1\cdot\hat{n}_2)$

($\gamma_0^2 - \beta^2\gamma_0^2 = 1$)

or $\quad 2(\gamma_0+1) = 2\alpha^2(1 - \hat{n}_1\cdot\hat{n}_2)$

or $\quad (\gamma_0+1) = \alpha^2(1 - \hat{n}_1\cdot\hat{n}_2)$

From (3) $\quad \alpha = \gamma_0 + 1$, $\therefore$

$$(\gamma_0+1) = \frac{1}{4}(\gamma_0+1)^2 (1 - \hat{n}_1 \cdot \hat{n}_2)$$

$$\frac{4}{\gamma_0+1} = 1 - \hat{n}_1 \cdot \hat{n}_2$$

$$\hat{n}_1 \cdot \hat{n}_2 = 1 - \frac{4}{\gamma+1}$$

opening angle

$$\boxed{\hat{n}_1 \cdot \hat{n}_2 = \frac{\gamma_0 - 3}{\gamma_0 + 1}} \quad (5)$$

If $\gamma_0 \to 1$ (annihilation at rest)

$\hat{n}_1 \cdot \hat{n}_2 \to -1$ (back to back)

each photon has $E = mc^2$

e.g. $\gamma_0 = 3$ ; $\hat{n}_1 \cdot \hat{n}_2 = 0$ $90^\circ$

$2E = mc^2(\gamma_0+1)$ ; $E = 2mc^2$

Most annihilation is at rest

e.g. $\gamma_0 \to \infty$

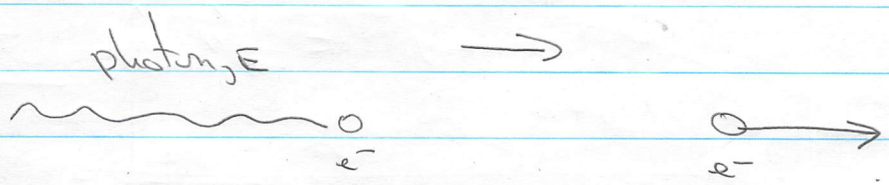
$\hat{n}_1 \cdot \hat{n}_2 = 1$ fwd direction

Show the beam stopper pictures

## The power of conservation laws

- allows us to eliminate certain interactions

e.g. Can




happen?

e⁻ at rest

Try. $C_0 E \Rightarrow$ $E + mc^2 = mc^2\gamma$ (6)

$C_0 \vec{M} \Rightarrow$ $\frac{E}{c} = mc\beta\gamma$ (7)

(6)/$mc^2$ $\Rightarrow$ $\alpha + 1 = \gamma$ (8)

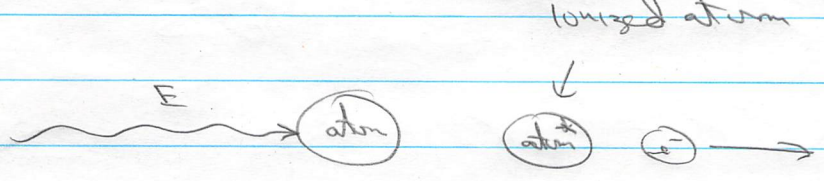
(7)/$mc$ $\Rightarrow$ $\alpha = \beta\gamma$ (9)

$(8)^2 - |(9)|^2 = (\alpha+1)^2 - \alpha^2 = 1$

$2\alpha + 1 = 1$

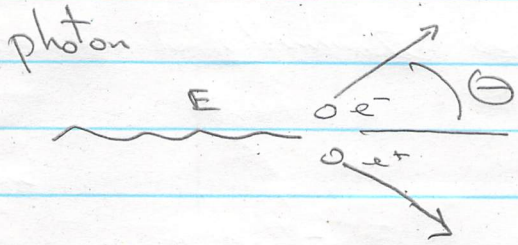
$\Rightarrow \alpha = 0$ no!

## Photoelectric effect?




The atom soaks up momentum, allowing this process to occur

Can




happen?

Try.

CoE $\quad E = 2mc^2\gamma$ (8) $\quad e^\pm$ pair must

CoM $\quad \frac{E}{c}\hat{n} = mc\gamma(\vec{\beta}_1 + \vec{\beta}_2)$ (9) each have the same energy

(8)/$mc^2$ $\Rightarrow$ $\alpha - \gamma = \gamma$ (10)

(9)/$mc$ $\Rightarrow$ $\alpha\hat{n} - \gamma\vec{\beta}_1 = \vec{\beta}_2\gamma$ (11)

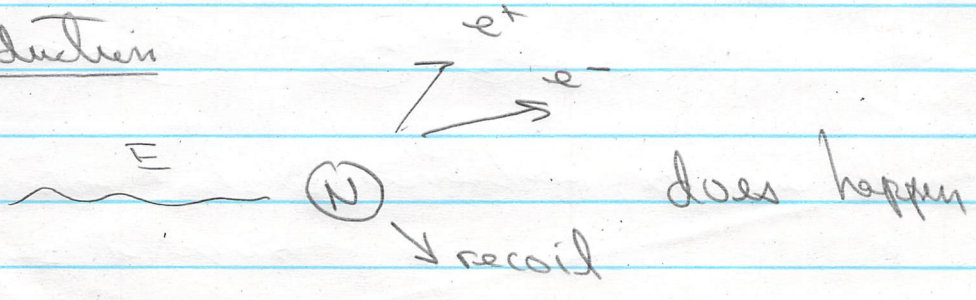
(10)$^2$ $-$ |(11)|$^2$ $\Rightarrow$ $\alpha^2 - 2\alpha\gamma + \gamma^2 - (\alpha^2 - 2\alpha\gamma\beta\cos\theta + \gamma^2\beta^2) = \gamma^2$

(2's cancel) $\quad \alpha\gamma\beta\cos\theta = \alpha\gamma$

$\cos\theta = \frac{1}{\beta} \Rightarrow \cos\theta > 1$

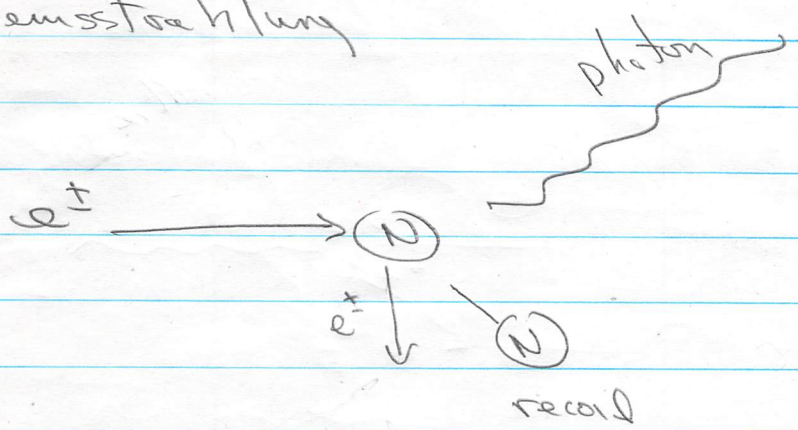
not possible

Pair production




does happen

Bremsstrahlung




needs a nucleus to conserve $E$ & $\vec{M}$

# A catalog of electron interactions

incident positron

(dominates at low energy)

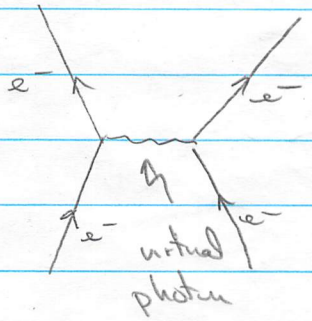



↑ time

$e^+$'s are $e^-$'s going backward in time

incident electron

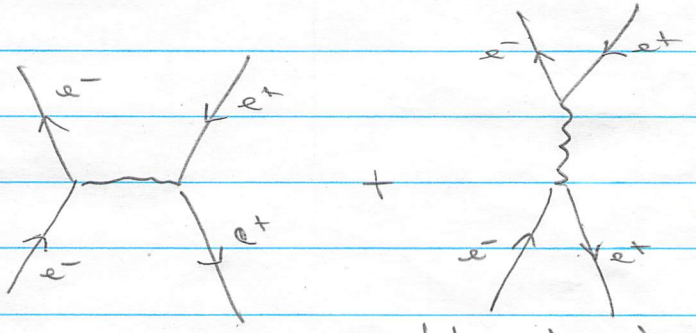
(dominates at Medium energy)




Møller interaction

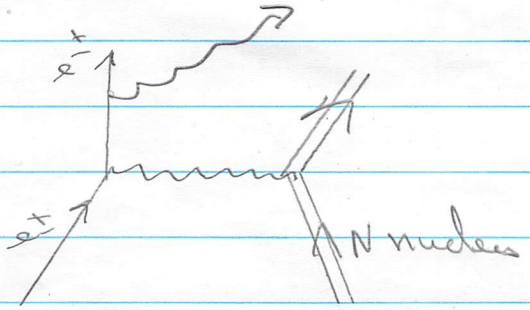
incident positron

(Medium energy)




Bhabha interaction

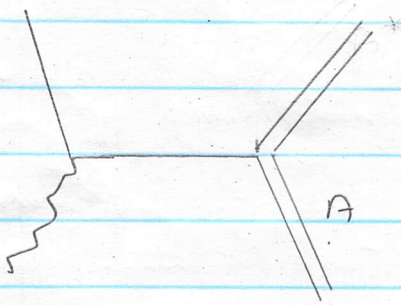
incident $e^-$ or $e^+$




Bremsstrahlung

# A catalog of photon interactions

low energy



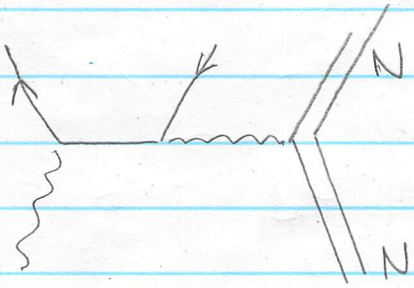

Photoelectric effect

medium energy



Compton

pair creation (high energy)




Pair creation

A philosophical question:

What is a photon?

Examine from a CoE/CoM perspective

Can 


occur?

CoE: $E_0 = E_1 + E_2$

CoM $\vec{}$ $\frac{E_0 \hat{n}}{c} = \frac{E_1 \hat{n}}{c} + \frac{E_2 \hat{n}}{c}$

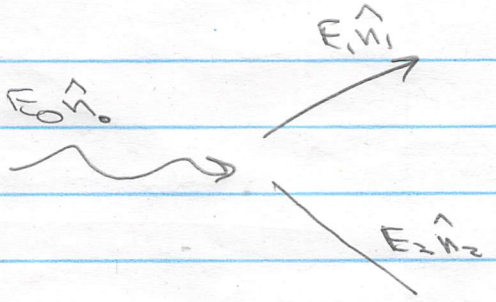
yes! as long as $E_0 = E_1 + E_2$

In fact, we can split it into as many pieces as we want, as long as

$$E_0 = \sum_{i=1}^{N} E_i$$

We can even let $N \to \infty$!

Can




occur?

Cons $E_0 = E_1 + E_2$ (1)

Cons $\vec{M}$ $\frac{E_0}{c}\hat{n}_0 = \frac{E_1}{c}\hat{n}_1 + \frac{E_2}{c}\hat{n}_2$ (2)

$(1)^2 - |c(2)|^2 \Rightarrow$

$0 = E_1 E_2 (1 - \hat{n}_1 \cdot \hat{n}_2)$

$\Rightarrow$ $E_1$ or $E_2$ is 0 (no split)

or

$\hat{n}_1 \cdot \hat{n}_2 = 1$ (provides cone)

So, a photon can be thought of as an arbitrary collection of lower energy photons, all traveling in the same direction

The Compton interaction can be thought of as a process that extracts a fraction of them and re-directs them

So, is a photon a wave? [YES]
Is it a particle? [YES]
but it is a divisible particle.

particle — carries energy
— " momentum
either divisible (photon)
or indivisible

∴ massless fundamental particle is co-by divisible

∴ fundamental particle with mass isn't
(It can decay into other particles, but that's
not the same as division)